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New Characterizations of Ratio Asymptotics for Orthogonal Polynomials[1]

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In this paper some new characterizations of ratio asymptotics for orthogonal polynomials are given.

Let α be a nondecreasing function on $[-1, 1]$ with infinitely many points of increase such that all moments of $d\alpha$ are finite and $\{P_n\}$,

$$P_n(x) := P_n(d\alpha, x) = \gamma_n x^n + \cdots \tag{1}$$

($\gamma_n := \gamma_n(d\alpha) > 0$), the orthonormal polynomials with respect to $d\alpha$. We call $d\alpha$ a measure.

Denote by $x_{kn} = x_{kn}(d\alpha)$ the zeros of $P_n(d\alpha)$ and by

$$L_n(f, x) := \sum_{k=1}^{n} f(x_{kn})\, \ell_{kn}(d\alpha, x),$$

the Lagrange interpolating polynomial of $f \in C[-1, 1]$, where the fundamental polynomials

$$\ell_{kn}(d\alpha, x) = \frac{P_n(d\alpha, x)}{P_n'(d\alpha, x_{kn})(x - x_{kn})}, \qquad k = 1, 2, ..., n.$$

As we know,

$$\lambda_n(x) := \lambda_n(d\alpha, x) = \left[\sum_{k=0}^{n-1} P_k^2(d\alpha, x)\right]^{-1} = \left[\sum_{k=1}^{n} \frac{\ell_{kn}^2(d\alpha, x)}{\lambda_{kn}}\right]^{-1}$$

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are called the Christoffel functions, where

$$\lambda_{kn} := \lambda_{kn}(d\alpha) = \lambda_n(d\alpha, x_{kn}), \qquad k = 1, 2, ..., n.$$

The orthogonal polynomials $\{P_n(d\alpha)\}$ satisfy the three-term recurrence relation

$$(x - \alpha_n(d\alpha))\, P_n(d\alpha, x) = \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)}\, P_{n+1}(d\alpha, x) + \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, P_{n-1}(d\alpha, x), \qquad n = 0, 1, ..., \tag{2}$$

where $P_{-1} = 0$, $\gamma_{-1} = 0$, and

$$\alpha_n(d\alpha) = \int_{-1}^{1} x P_n^2(d\alpha, x)\, d\alpha(x). \tag{3}$$

In his memoir [2], Nevai introduced the class of measures $\mathbf{M} := \mathbf{M}(0, 1)$, for which $d\alpha \in \mathbf{M}$ means

$$\lim_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} = \frac{1}{2} \tag{4}$$

and

$$\lim_{n\to\infty} \alpha_n(d\alpha) = 0. \tag{5}$$

Nevai proved the following characterizations of orthogonal polynomials with respect to measures in $\mathbf{M}$.

THEOREM A [2, Theorems 4.1.12 and 4.1.13, pp. 32–34]. *Let $d\alpha$ be a measure supported in $[-1, 1]$. If $d\alpha \in \mathbf{M}$, then*

$$\lim_{n\to\infty} \frac{P_n(d\alpha, z)}{P_{n-1}(d\alpha, z)} = \phi(z) \tag{6}$$

holds uniformly for every closed subset of $\mathbb{C} \setminus [-1, 1]$, where $\phi(z) = z + \sqrt{z^2 - 1}$.

Conversely, if (6) *holds for an unbounded sequence of values of z, then $d\alpha \in \mathbf{M}$.*

THEOREM B [2, Theorems 3.2.3 and 3.2.4, pp. 17–34]. *Let $d\alpha$ be a measure supported in $[-1, 1]$. Then $d\alpha \in \mathbf{M}$ if and only if*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, P_{n-1}^2(d\alpha, x_{kn}) = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{1 - x^2}\, dx \tag{7}$$

holds for every bounded and Riemann integrable function f on $[-1, 1]$.

The main aim of this paper is to give new characterizations of orthogonal polynomials with respect to measures in **M**, which are stated as follows.

THEOREM 1. *Let $d\alpha$ be a measure supported in $[-1, 1]$. If $d\alpha \in \mathbf{M}$, then the relation* (6) *holds for every $z \in \mathbb{C} \setminus (-1, 1)$ and*

$$\begin{aligned} &\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{P_{n-1}^2(d\alpha, x_{kn})}{1-x_{kn}^2} \\ &\quad = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{P_{n-1}^2(d\alpha, x_{kn})}{1-x_{kn}} \\ &\quad = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{P_{n-1}^2(d\alpha, x_{kn})}{1+x_{kn}} \\ &\quad = 2. \end{aligned} \tag{8}$$

In particular, we have

$$\lim_{n\to\infty} \frac{P_n(d\alpha, 1)}{P_{n-1}(d\alpha, 1)} = -\lim_{n\to\infty} \frac{P_n(d\alpha, -1)}{P_{n-1}(d\alpha, -1)} = 1. \tag{9}$$

THEOREM 2. *Let $d\alpha$ be a measure supported in $[-1, 1]$. Then the following statements are equivalent:*

(a) *the relation $d\alpha \in \mathbf{M}$ holds;*

(b) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{P_{n-1}^2(d\alpha, x_{kn})}{1-x_{kn}^2} = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \tag{10}$$

holds for every bounded and Riemann integrable function f on $[-1, 1]$;

(c) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{P_{n-1}^2(d\alpha, x_{kn})}{1-x_{kn}} = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{\frac{1+x}{1-x}}\, dx \tag{11}$$

holds for every bounded and Riemann integrable function f on $[-1, 1]$;

(d) *the relation*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn}) \frac{P_{n-1}^2(d\alpha, x_{kn})}{1+x_{kn}} = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{\frac{1-x}{1+x}}\, dx \tag{12}$$

holds for every bounded and Riemann integrable function f on $[-1, 1]$.

THEOREM 3. *Let $d\alpha \in \mathbf{M}$, f be a bounded and Riemann integrable function on $[-1, 1]$, U_n be the nth Chebyshev polynomial of the second kind, and m be a fixed integer. Then*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, P_{n-1}(d\alpha, x_{kn}) \frac{P_{n+m}(d\alpha, x_{kn})}{1-x_{kn}^2} = -\frac{\operatorname{sign} m}{\pi} \int_{-1}^{1} f(x)\, U_{|m|-1}(x) \frac{dx}{\sqrt{1-x^2}}, \tag{13}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, P_{n-1}(d\alpha, x_{kn}) \frac{P_{n+m}(d\alpha, x_{kn})}{1-x_{kn}} = -\frac{\operatorname{sign} m}{\pi} \int_{-1}^{1} f(x)\, U_{|m|-1}(x) \sqrt{\frac{1+x}{1-x}}\, dx, \tag{14}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, P_{n-1}(d\alpha, x_{kn}) \frac{P_{n+m}(d\alpha, x_{kn})}{1+x_{kn}} = -\frac{\operatorname{sign} m}{\pi} \int_{-1}^{1} f(x)\, U_{|m|-1}(x) \sqrt{\frac{1-x}{1+x}}\, dx. \tag{15}$$

The relations (10) and (13) were proved by Nevai in [2, Theorem 4.2.3, pp. 39–41; Theorem 4.2.17, p. 48] for the Szegő class $\mathbf{S}$, for which $d\alpha \in \mathbf{S}$ means $[\ln \alpha'(x)]/\sqrt{1-x^2} \in L^1(-1, 1)$.

Before proving the theorems we establish an auxiliary result which will play a crucial role in this paper.

LEMMA 1. *If a sequence of positive numbers $\{a_n\}_{n=1}^{\infty}$ satisfies*

$$\lim_{n\to\infty} \frac{a_{n-1}+a_{n+1}}{a_n} = 2, \tag{16}$$

then

$$\lim_{n\to\infty} \frac{a_{n-1}}{a_n} = 1. \tag{17}$$

Proof. First we observe that

$$\frac{1}{2} \leqslant A := \liminf_{n\to\infty} \frac{a_{n-1}}{a_n} \leqslant B := \limsup_{n\to\infty} \frac{a_{n-1}}{a_n} \leqslant 2, \tag{18}$$

for otherwise it would contradict (16). Next it suffices to show

$$A = B = 1. \tag{19}$$

To prove $A = 1$ assume that for some subsequence of positive integers $\{n_k\}$ we have

$$\lim_{k\to\infty} \frac{a_{n_k-1}}{a_{n_k}} = A. \tag{20}$$

Since by (16)

$$\lim_{k\to\infty} \frac{a_{n_k-2}+a_{n_k}}{a_{n_k-1}} = \lim_{k\to\infty} \frac{a_{n_k-1}+a_{n_k+1}}{a_{n_k}} = 2, \tag{21}$$

according to (20) we obtain

$$\lim_{k\to\infty} \frac{a_{n_k-2}}{a_{n_k-1}} = 2 - \lim_{k\to\infty} \frac{a_{n_k}}{a_{n_k-1}} = 2 - \frac{1}{A} = \frac{2A-1}{A}.$$

By (18) we have

$$\frac{2A-1}{A} \geqslant A.$$

Hence $1 + A^2 - 2A \leqslant 0$, i.e., $(A-1)^2 \leqslant 0$. This gives $A = 1$. Similarly, if

$$\lim_{k\to\infty} \frac{a_{n_k-1}}{a_{n_k}} = B,$$

then by the same argument as above we obtain $B = 1$. This proves (19) and (17). ▮

Proof of Theorem 1. To prove that the relation (6) holds for every $z \in \mathbb{C} \setminus (-1, 1)$ by Theorem A it is enough to show the relation (9). Now let us do it. The relation (2) with $x = 1$ gives

$$1 - \alpha_n = \frac{\gamma_n P_{n+1}(1)}{\gamma_{n+1} P_n(1)} + \frac{\gamma_{n-1} P_{n-1}(1)}{\gamma_n P_n(1)}. \tag{22}$$

By (22) we obtain

$$\begin{aligned} \frac{P_{n-1}(1)+P_{n+1}(1)}{P_n(1)} &= \left[\frac{P_{n-1}(1)}{P_n(1)} + \frac{\gamma_n^2 P_{n+1}(1)}{\gamma_{n-1}\gamma_{n+1} P_n(1)}\right] + \left[1 - \frac{\gamma_n^2}{\gamma_{n-1}\gamma_{n+1}}\right] \frac{P_{n+1}(1)}{P_n(1)} \\ &= \frac{\gamma_n}{\gamma_{n-1}}(1-\alpha_n) + \left[1 - \frac{\gamma_n^2}{\gamma_{n-1}\gamma_{n+1}}\right] \frac{P_{n+1}(1)}{P_n(1)}. \end{aligned}$$

By (3) we see $|\alpha_n| \leqslant 1$ and hence $0 \leqslant 1-\alpha_n \leqslant 2$. By means of (22) we get

$$\frac{P_{n+1}(1)}{P_n(1)} \leqslant \frac{\gamma_{n+1}}{\gamma_n}(1-\alpha_n).$$

Hence by (4) and (5)

$$\lim_{n\to\infty} \frac{P_{n-1}(1)+P_{n+1}(1)}{P_n(1)} = \lim_{n\to\infty} \left\{ \frac{\gamma_n}{\gamma_{n-1}}(1-\alpha_n) + \left[1 - \frac{\gamma_n^2}{\gamma_{n-1}\gamma_{n+1}}\right] \frac{P_{n+1}(1)}{P_n(1)} \right\} = 2.$$

Applying Lemma 1 we get

$$\lim_{n\to\infty} \frac{P_n(1)}{P_{n-1}(1)} = \lim_{n\to\infty} \frac{P_{n-1}(1)}{P_n(1)} = 1.$$

Similarly, the second relation of (9) follows from the relation (2) with $x=-1$:

$$1+\alpha_n = \frac{\gamma_n}{\gamma_{n+1}} \frac{|P_{n+1}(-1)|}{|P_n(-1)|} + \frac{\gamma_{n-1}}{\gamma_n} \frac{|P_{n-1}(-1)|}{|P_n(-1)|}. \tag{23}$$

To prove (8) we need to use some known formulas [1]:

$$\frac{P_{n-1}(1)}{P_n(1)} = \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_{kn} \frac{P_{n-1}^2(x_{kn})}{1-x_{kn}}, \tag{24}$$

$$\frac{P_{n-1}(-1)}{P_n(-1)} = -\frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_{kn} \frac{P_{n-1}^2(x_{kn})}{1+x_{kn}}, \tag{25}$$

and

$$\frac{P_{n-1}(1)}{P_n(1)} - \frac{P_{n-1}(-1)}{P_n(-1)} = \frac{2\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_{kn} \frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}. \tag{26}$$

Clearly, (8) follows from (9), (4), and (24)–(26). ∎

Proof of Theorem 2. We give the proof of equivalence of Statements (a) and (b) only; the proof of equivalence of Statements (a) and (c) as well as Statements (a) and (d) is similar. The proof follows the line given in [2, pp. 40–41].

(a) ⇒ (b). Let ε, $0<\varepsilon<1$, be an arbitrary and fixed number. Since the function $[f(x)/(1-x^2)]\, I(x)_{[-1+\varepsilon,\, 1-\varepsilon]}$ is bounded and Riemann integrable on $[-1, 1]$, where $I_{[-1+\varepsilon,\, 1-\varepsilon]}$ is the characteristic function of the interval $[-1+\varepsilon, 1-\varepsilon]$, by Theorem B we have

$$\lim_{n\to\infty} \sum_{|x_{kn}|\leqslant 1-\varepsilon} \lambda_{kn} f(x_{kn}) \frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2} = \frac{2}{\pi} \int_{-1+\varepsilon}^{1-\varepsilon} \frac{f(x)}{\sqrt{1-x^2}}\, dx. \tag{27}$$

Inserting $f=1$ into (27) gives

$$\lim_{n\to\infty}\sum_{|x_{kn}|\leqslant 1-\varepsilon}\lambda_{kn}\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}=\frac{2}{\pi}\int_{-1+\varepsilon}^{1-\varepsilon}\frac{1}{\sqrt{1-x^2}}\,dx. \tag{28}$$

Using an obvious formula

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx=1$$

it follows from (8) and (28) that

$$\lim_{n\to\infty}\sum_{|x_{kn}|>1-\varepsilon}\lambda_{kn}\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}=\frac{4}{\pi}\int_{1-\varepsilon}^{1}\frac{1}{\sqrt{1-x^2}}\,dx. \tag{29}$$

We have

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\lambda_{kn}f(x_{kn})\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}-\frac{2}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx\right|\\
&\quad=\left|\left[\sum_{|x_{kn}|\leqslant 1-\varepsilon}\lambda_{kn}f(x_{kn})\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}-\frac{2}{\pi}\int_{-1+\varepsilon}^{1-\varepsilon}\frac{f(x)}{\sqrt{1-x^2}}\,dx\right]\right.\\
&\qquad+\sum_{|x_{kn}|>1-\varepsilon}\lambda_{kn}f(x_{kn})\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}\\
&\qquad\left.-\frac{2}{\pi}\left[\int_{-1}^{-1+\varepsilon}\frac{f(x)}{\sqrt{1-x^2}}\,dx+\int_{1-\varepsilon}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx\right]\right|\\
&\quad\leqslant\left|\sum_{|x_{kn}|\leqslant 1-\varepsilon}\lambda_{kn}f(x_{kn})\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}-\frac{2}{\pi}\int_{-1+\varepsilon}^{1-\varepsilon}\frac{f(x)}{\sqrt{1-x^2}}\,dx\right|\\
&\qquad+\left[\sup_{-1\leqslant x\leqslant 1}|f(x)|\right]\left[\sum_{|x_{kn}|>1-\varepsilon}\lambda_{kn}\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}+\frac{4}{\pi}\int_{1-\varepsilon}^{1}\frac{1}{\sqrt{1-x^2}}\,dx\right].
\end{aligned}$$

Then by virtue of (27) and (29)

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\sum_{k=1}^{n}\lambda_{kn}f(x_{kn})\frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}-\frac{2}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx\right|\\
&\qquad\leqslant\left[\sup_{-1\leqslant x\leqslant 1}|f(x)|\right]\frac{8}{\pi}\int_{1-\varepsilon}^{1}\frac{1}{\sqrt{1-x^2}}\,dx.
\end{aligned}$$

Hence as $\varepsilon\to 0$ we get (10).

(b) $\Rightarrow$ (a). If f is an arbitrary bounded and Riemann integrable function on $[-1, 1]$, then $f(x)(1-x^2)$ is also a bounded and Riemann integrable function on $[-1, 1]$. Inserting the function $f(x)(1-x^2)$ into (10) yields the relation (7). Applying Theorem B we conclude $d\alpha \in \mathbf{M}$. ▮

Proof of Theorem 3. Again we show (13) only, because the proof of (14) and (15) runs in a similar way. Applying Theorems 3.1.3 ($m > 0$) and 3.1.13 ($m < 0$) in [2, pp. 9 and 13] and using the recurrence relation (2) we get

$$P_{n+m}(x_{kn}) = -[\operatorname{sign} m]\, U_{|m|-1}(x_{kn})\, P_{n-1}(x_{kn}) + c_n[|P_{n-1}(x_{kn})| + |P_{n-2}(x_{kn})|],$$

where $\lim_{n\to\infty} c_n = 0$ holds uniformly for $1 \leqslant k \leqslant n$ if m is fixed. Using the recurrence relation (2) replacing n by $n-1$ and putting $x = x_{kn}$ we obtain

$$|P_{n-2}(x_{kn})| = \left|\frac{\gamma_{n-1}}{\gamma_{n-2}}(x_{kn} - \alpha_{n-1})\, P_{n-1}(x_{kn})\right| \leqslant \frac{2\gamma_{n-1}}{\gamma_{n-2}}\, |P_{n-1}(x_{kn})|.$$

Thus

$$P_{n+m}(x_{kn}) = -[\operatorname{sign} m]\, U_{|m|-1}(x_{kn})\, P_{n-1}(x_{kn}) + c_n^*\, |P_{n-1}(x_{kn})|,$$

where $\lim_{n\to\infty} c_n^* = 0$ again holds uniformly for $1 \leqslant k \leqslant n$ if m is fixed. Hence

$$\sum_{k=1}^{n} \lambda_{kn} f(x_{kn})\, P_{n-1}(x_{kn}) \frac{P_{n+m}(x_{kn})}{1-x_{kn}^2}$$
$$= -[\operatorname{sign} m] \sum_{k=1}^{n} \lambda_{kn} f(x_{kn})\, U_{|m|-1}(x_{kn}) \frac{P_{n-1}^2(x_{kn})}{1-x_{kn}^2}$$
$$+ c_n^* \sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) \frac{P_{n-1}(x_{kn})\, |P_{n-1}(x_{kn})|}{1-x_{kn}^2},$$

which by (10) implies (13). ▮

REFERENCES

1. G. Freud, On Hermite–Fejér interpolation processes, *Studia Sci. Math. Hungar.* **7** (1972), 307–316.
2. P. Nevai, "Orthogonal Polynomials," Memoirs of the Amer. Math. Soc., Vol. 213, Amer. Math. Soc., Providence, 1979.